SPHERICALLY SYMMETRIC STEFAN PROBLEM WITH A BOUNDARY CONDITION OF THE THIRD KIND

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A nonlinear integrodifferential equation is derived for the temperature distribution in the region under study. The determination of the time for the unknown boundary to reach a given position is reduced to a quadrature.

The method of successive approximations is used in [1, 2] to solve the two-dimensional and axisymmetric Stefan problem without initial conditions. We develop a method for solving the spherically symmetric Stefan problem with a boundary condition of the third kind.

After introducing appropriate dimensionless variables the problem can be formulated mathematically in the form

$$\frac{\partial U}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x} x^2 \frac{\partial U}{\partial x} \quad \text{in } D, \tag{1}$$

$$\frac{\partial U}{\partial x} = \alpha U \quad \text{for} \quad x = 1, \tag{2}$$

$$U = \beta \quad \text{for} \quad x = \Delta(t), \tag{3}$$

$$\frac{d\Delta}{dt} = \frac{\partial u}{\partial x} \quad \text{for} \quad x = \Delta(t), \tag{4}$$

$$\Delta(0) = 1, \tag{5}$$

when $D = \{x,\ t;\ 1 < x < \Delta(t);\ 0 < t < \infty\}$ is an open domain in which the solution of the problem must be constructed.

Problem (1)-(5) corresponds to an ice covered sphere whose surface is cooled by convection, with the ambient temperature at zero time equal to the temperature of the phase transition. The required functions are the dimensionless temperature U(x, t) and the position of the phase transition boundary $x = \Delta(t)$. The solution of problem (1)-(5) is constructed as in [2].

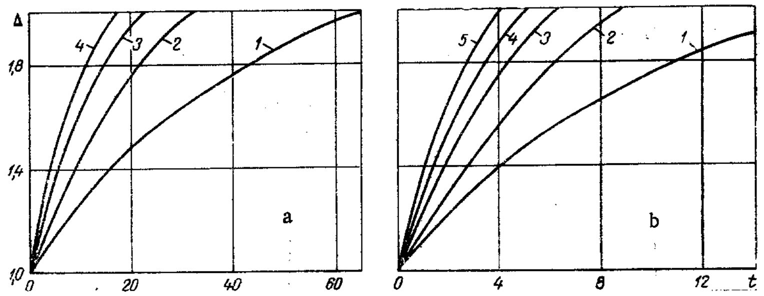

Fig. 1. Position of phase transition boundary: H = 0.05. a: 1) β = 0.8; 2) 0.6; 3) 0.4; 4) 0.2; α = 0.2; b: 1) α = 0.2; 2) 0.4; 3) 0.6; 4) 0.8; 5) 1.0; β = 0.8.

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TABLE 1. Time to Build Up a Layer of Ice on a Convectively Cooled Spherically Symmetric Surface

α, β	H	Δ: 1,05	1,1	1,15	1,2	1,25	1,3	1 35	1,4	1,45	1,5
α=0,4	0,05	0,185045	0,407337	0,647545	0,907327	1,187965	1,490742	1,816929	2,167785	2,544559	2,948486
	0,025	0,194953	0,416822	0,656787	0,916208	1,196429	1,498764	1,824507	1,174931	2,551290	2,954823
β=0,8	0,0125	0,199715	0,411239	0,660886	0,920019	1,199975					
	0,00625	0,202073	0,423425								
	0,05	0,166864	0,369850	0,590096	0,829231	1,088577	1,369442	1,673125	2,000913	2,354078	2,733878
α=0,6	0,025	0,176266	0,378426	0,598145	0,836749	1,095566	1,375914	1,679102	2,006420	2,359143	2,738529
	0,0125	0,181354	0,383657	0,603338	0,841820	1,100448					
β=0,6	0,00625	0,183658	0,385767								
	0,05	0,129314	0,290123	0,464524	0,653995	0,859666	1,082644	1,324033	1,584932	1,866430	2,169604
α=0,8	0,025	0,139175	0,298205	0,472178	0,661210	0,866412	1,088931	1,329865	1,590329	1,871414	2,174200
	0,0125	0,142491	0,302340	0,476108	0,664914	0,869895					
β=0,6	0,00625	0,144679	0,304440								

The problem is solved

$$U = \frac{\beta\left[1+\alpha\left(1-\frac{1}{x}\right)\right]}{1+\alpha\left(1-\frac{1}{\Delta}\right)} + AU; \tag{6}$$

$$t = BU, \tag{7}$$

where the operators A and B act on the function U(x, Δ) according to the rule

$$AU = \left\{\alpha\beta\left[\left[\int_1^x \frac{1}{x^2}\int_1^x x^2\frac{\partial U}{\partial \Delta}\,dxdx - \frac{1+\alpha\left(1-\frac{1}{x}\right)}{1+\alpha\left(1-\frac{1}{\Delta}\right)}\right.\right.\right.$$

$$\left.\left.\times\int_1^\Delta \frac{1}{x^2}\int_1^x x^2\frac{\partial U}{\partial \Delta}\,dxdx\right]\right]\Big/\left\{\left[1+\alpha\left(1-\frac{1}{\Delta}\right]\right)\right.$$

$$\left.\times\left(\Delta^2-\int_1^\Delta x^2\frac{\partial U}{\partial \Delta}dx\right)+\alpha\int_1^\Delta\frac{1}{x^2}\int_1^x x^2\frac{\partial U}{\partial \Delta}\,dxdx\right]\right\}, \tag{8}$$

$$BU = \frac{1}{\alpha\beta}\int_1^\Delta\left\{\left[1+\alpha\left(1-\frac{1}{\Delta}\right]\right)\left(\Delta^2-\int_1^\Delta x^2\frac{\partial U}{\partial \Delta}dx\right)\right.$$

$$\left.+\alpha\int_1^\Delta\frac{1}{x^2}\int_1^x x^2\frac{\partial U}{\partial \Delta}\,dxdx\right\}d\Delta. \tag{9}$$

The integrodifferential equation (6) can be solved by the method of successive approximations, and then by using (7) the time for a layer of ice of thickness Δ to build up on the surface of the sphere can be determined.

The iteration scheme has the form

$$U_{k+1} = \frac{\beta\left[1+\alpha\left(1-\frac{1}{x}\right)\right]}{1+\alpha\left(1-\frac{1}{\Delta}\right)} + AU_k;\quad t_{k+1} = BU_k; \tag{10}$$

$$U_0 = \frac{\beta\left[1+\alpha\left(1-\frac{1}{x}\right)\right]}{1+\alpha\left(1-\frac{1}{\Delta}\right)};\quad \frac{d\Delta_0}{dt} = \frac{\alpha\beta}{\Delta_0^2\left[1+\alpha\left(1-\frac{1}{\Delta_0}\right]\right)}. \tag{11}$$

The zero approximation corresponds to the case of an infinitely small specific heat of the medium.

The criterion for the final iteration was the condition $|U_{k+1}(x, t)-U_k(x, t)| < 10^{-4}$ uniformly in domain D for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$; $\beta = 0.2, 0.4, 0.6, 0.8$. There were no more than four iterations. The character of the change $\Delta(t)$ for a change in the mesh size H = 0.05, 0.025, 0.0125, 0.00625 is shown in Table 1.

Figures 1a and b show the solutions for various values of α and β and H = 0.05.

In conclusion we note that Eqs. (6) and (7) are more convenient than (1)-(5) for solving the problem in the finite difference formulation.

LITERATURE CITED

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2. L. A. Vygovskaya and A. M. Makarov, Prikl. Mekhan. i Tekh. Fiz., 6, 182 (1971).